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Infinite Loops in Finite Time: Some Observations

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Abstract

Formulating physical theories using real-valued space and time often raises the difficult issue of *clustered variation:* states that change their value infinitely often in a finite interval, or infinite sequences of events that occur in a finite interval. Some physical theories permit nonsensical models unless clustered variation is prohibited; others force clustered variation to occur. This paper surveys a number of technical issues involved with clustered variation, without attempting to find a uniform treatment of the issue. We present examples of theories where clustered variation must be allowed and of theories where it must be prohibited. We discuss the meaning of plans containing infinite series of events. We present constraints on real-valued fluents and spatial fluents that guarantee discrete variation of important associated states. We show that requiring discrete variation may be inconsistent with allowing infinitely many objects in the universe, even if each object is individually well-behaved. We show how the need for requiring discrete variation may depend subtly on the particular ontology chosen for the domain.

1 Introduction

Viewed from a modern perspective, the great insight underlying the paradoxes of Zeno the Eleatic is that the commonsense understanding of time and space is profoundly confused on the subject of their infinite divisibility. On the one hand, it seems absurd to suppose that two instants or two points can be distinct and yet have no instant or point between them. On the other hand, if space and time are dense, then the arrow trying to reach its goal or Achilles trying to catch up with the tortoise must execute infinitely many movements, which does not seem right either.

Twenty-five hundred years of subsequent mathematics and philosophy have pretty well settled the mathematical and physical problems here. The consensus after millenia is that, for most practical purposes, one may take space and time to be dense and complete (i.e. $\Re^n$) and that it is consistent to say both, "Achiles executes the action 'Go to the current position of the tortoise' infinitely many times," and "Achilles catches up to the tortoise in finite time." This theory leads to paradoxes even stranger than Zeno's: the expression of a discontinuous function as the infinite sum of continuous functions; space-filling curves; nowhere-differentiable continuous curves; the Banach-Tarski paradox; and so on. Some of these, such as Fourier series, have been accommodated within the language of physics; others, such as the Banach-Tarski paradox, are characterized as mathematical constructs without physical realization. The dividing line between the two is generally left to the

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physical intuitions of the human understander. These intuitions change over time, of course. In the nineteenth century, nowhere-differentiable curves were mathematical monstrosities; today, they are an important element of our descriptive language in a whole range of applications.

All these problems come up again when we try to characterize commonsense domains for automated reasoners. A theory of a domain that uses real-valued time must prescribe which of the mathematically possible behaviors can, in fact, occur within the domain. Particularly knotty problems are associated with *clustered variation* or *chatter*: a Boolean fluent that change its truth-value infinitely often in a finite time interval, or an infinite sequences of events that occurs within a finite time interval. For example, should a theory allow a light to be on during time instants corresponding to rational numbers and off during time instants corresponding to irrational numbers? Should it allow a light to be on during the interval $t \in [0, 1/2)$, off during $t \in [1/2, 3/4)$, on during $t \in [3/4, 7/8)$, off during $t \in [7/8, 15/16) \ldots$? Similarly, should it allow the execution of an infinite loop in which the first iteration is executed in 1/2 second, the second in 1/4 second, the third in 1/8, and so on?[1]

A number of researchers (e.g. [Hayes, 85], [Fleck, 88] [Kaufmann, 91]), noting that such clustering can neither be perceived, due to the limited resolution of human perception, nor actually occur, due to the atomic and quantized nature of very small-scale physics, have suggested that $\Re^n$ is a poor choice as a model for space and time, and that some other topology be found. However, the alternatives that have been suggested seem to lead to constructs as peculiar as those encountered in $\Re^n$ (see, for example, the definition of a "straight line" in [Kaufmann, 91]) and there are many basic physical phenomena, such as rotation, that have not received any adequate treatment in these theories. (Fleck's proposed topology, which uses $\Re^n$ as a substrate, may be an exception. However, it is not clear how her approach can be applied to physical theories.) Therefore, it seems to me that the suggestion of alternative topologies must be taken as conjectural until someone can use them in fully developed physical theories. I do not think that the two motivations mentioned above are promising leads toward better theories of space and time. The issue of limited perceptual resolution, I believe, is best treated as an aspect of perception, rather than as an fundamental quality of space and time [Davis, 89a]. The quantized nature of small-scale physics involves phenomena that are much further from a commonsense understanding than the difficulties of real-valued space and time.[2]

An alternative policy would be to use real-valued space and time, but to impose well-behavedness conditions on physical entities. One attempt at such a policy was axiom 9 of [McDermott, 82] (proposed by me). Let us say that a Boolean fluent *varies discretely* if it nowhere exhibits clustered variation; that is, if it changes its value only finitely many times in any bounded time interval. (An equivalent, formally simpler, definition is given in Theorem 1 of the appendix.) Axiom 9 of [McDermott, 82], which we will here call the "axiom of discrete variation", posited that every Boolean fluent must vary discretely.[3]

[1] The use of real-valued space and time also raises other problems of small scale geometry, such as "Do solid objects occupy open or closed point sets?" "Can an solid object occupy a two dimensional surface? or a shape that joins itself at a point?" "Must the surface of an object be everywhere smooth? or almost everywhere smooth?" These problems are also very hard, and, it turns out, closely connected to the problems of clustered variation addressed here. Different from either of these is the phenomenon of stutter, discussed in [Forbus, 85], in which a physical system departs from dynamic equilibrium for an instant, and is immediately returns to the equilibrium state. This is not mathematically possible for continuous functions; rather, it constitutes a proof by reductio that the system must maintain the equilibrium state.

[2] It may be observed that some infinitary paradoxes have been of great importance in physics: Ohlber's paradox that the night sky should be infinitely bright; the classical calculation that a black body should radiate infinite energy; the infinite self-energy of a point electron; and so on. However, there is a critical difference between these paradoxes and those of clustered variation; namely, that the physical paradoxes remain problematic even if the quantities are not infinite but merely very large. If the universe is large enough, then the sky should be much brighter than it is, and if the classical account of thermal radiation is anywhere close to right then the radiation of a black body should be much larger than it is, and if an electron is small enough, then its mass would have to be much greater than it is. By contrast, clustered variation raises no problem until it becomes truly infinite; very frequent variation is not problematic.

[3] A similar but weaker rule was proposed in [Hamblin, 71]. [Galton, 87] discusses some philosophical treatments of

"The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil." [Russell, 19]. It soon became clear that imposing the axiom of discrete variation creates two serious difficulties. The first is that the axiom of discrete variation can only hold if certain restrictions are placed on the the language of states, the behavior of real-valued fluents, and the shapes, motions, and placements of objects. For example, "the temperature of the tea being a rational number" must be excluded as a possible state. The function $x(t) = t\sin(1/t)$ must be excluded as a possible behavior, if the condition $x > 0$ is to be allowed as a fluent. These restrictions were discussed in [McDermott, 82], but not in adequate depth.

The second difficulty is that a number of physical theories actually predict the occurrence of clustered variation. These theories are idealizations, but they are reasonable and useful idealizations, that can only be avoided at substantial cost. The best known of these (discussed in [Fleck, 88]) is the bouncing ball (Figure 1). The simplest and most straightforward model of a ball bouncing on the ground is that the ball is a rigid object, and that at each bounce, the ball bounces up with a speed that is a fixed fraction μ of the speed with which it hit.[4] Since the time between bounces is proportional to the speed leaving the ground, it follows that the time between bounces decreases in a geometric series, and that therefore the ball attains perfect rest in a finite time, having carried out infinitely many bounces. (Note that this is different from a standard damped harmonic oscillator, in which the frequency is constant, and the system therefore in principle never reaches a state of absolute rest.) Admittedly, this is an idealization; with an actual ball, beyond a certain point, the system will be dominated by the internal vibrations of the ball, which have a fixed frequency. However, to take this into account would require modelling the ball as an elastic object, which greatly complicates reasoning.

It is important to note that this prediction, though peculiar, does not constitute an *inconsistency* in the theory of rigid solid objects. The position and the velocity of the bouncing ball approach a unique limit as the bounces go to zero, so its state after the infinite series of bounces is well-defined. The scenario with infinitely many bounces does not contradict any of the laws of Newtonian physics.

A common reaction to this example is to propose that a reasoning system should treat the ball as rigid except at times where it is predicts clustered variation, where it should use a more realistic model. The problem is that in a general reasoning system, the prediction of clustered variation may be subtly hidden. Consider, for example, the following problem. Given that a ball is at rest on a surface at time t, is it possible that the ball was bouncing any time before t? (We assume that the ball and the surface are isolated from other interference.) It is not hard to imagine (or, for that matter, to build) a program that constructs the following chain of reasoning.

> Suppose that the ball was bouncing before t. Consider the last bounce B. It must have attained height x on B, and therefore hit the table with velocity $v = \sqrt{2gx}$. But then it would have bounced up again with velocity μv, which contradicts the assumption that B was the last bounce. Therefore, the ball has always been at rest on the table.

Note, first, that there is nothing obvious anomalous in the conclusion to trigger the suspicions of the program that something is wrong — nothing that would force the withdrawal of a default assumption, for example. Note, also, that the assumption of discrete variation is "hidden" here in the innocent-looking step of assuming that if it bounced, then there was a last bounce. Neither the program nor its creator need ever do any thinking about infinite sequences of events. Put the point another way: a qualitative simulator, addressing this system, might well generate a envisionment graph in which there was no transition from a bouncing state to a rest state (Figure 2). This graph is not incorrect; however, it would be a mistake to conclude from it that a ball that is now bouncing will never be at rest, or that a ball that is now at rest was never bouncing.

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clustered variation.

[4] Its velocity may be posited by convention to be the limit of the speed from previous times. [Davis, 88]

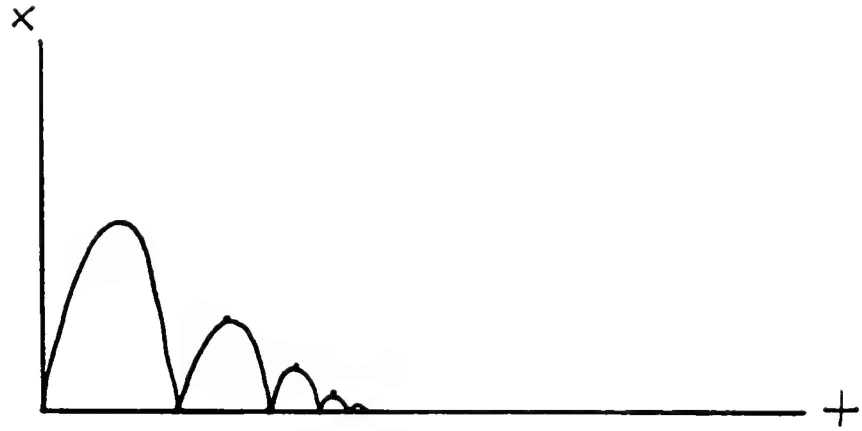


Figure 1: Trajectory of bouncing ball

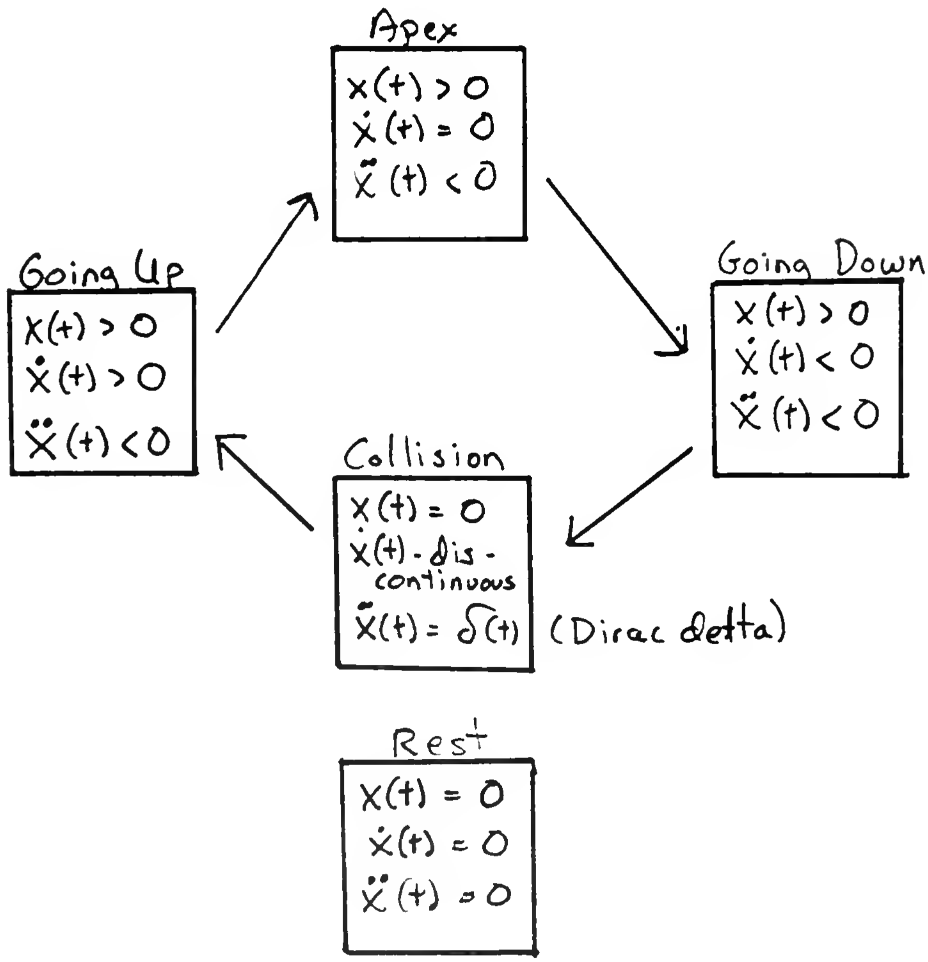


Figure 2: Envisionment for bouncing ball

It is clear, in retrospect, that the axiom of discrete variation was substantially overkill, in trying to apply to all fluents. A more reasonable approach is to define "discrete variation" as an interesting property, and then to posit that specific key fluents have this property. The problem, then, is to determine which fluents can and should be so constrained.

Over the last several years, I have worked on formal characterizations of a variety of physical theories using real-valued space and time (e.g. [Davis, 88, 90, 91]). Each time, the issue of clustered variation has arisen in new ways and created new and peculiar problems. My experience has been that there are no general well-behavedness conditions that can be imposed across the board on all theories of this kind. Rather, the issue must be thought through seperately for each such theory, and for each fluent in the theory. The kinds of well-behavedness conditions to be imposed depend delicately on the particular range of physical phenomena considered; the particular idealizations of these phenomena that are adopted; the kinds of fluents that are considered as first-class entities; and the way in which the theory addresses other issues of space and time.

Accordingly, this paper does not attempt a general discussion of the problem of clustered variation. Rather, it addresses a number of technical issues that arise. Section 2 discusses further why, aside from intuitive discomfort, one should want to exclude clustered variation in certain theories. Section 3 presents a few more examples of natural theories, in addition to the bouncing ball, that generate clustered variation. Section 4 discusses infinite sequences of events. Section 5 presents conditions on the types of states, real-valued fluents, shapes, and motions that are sufficient to guarantee discrete variation. Section 6 discusses how the necessity or impact of requiring discrete variation may differ depending on how the world is carved up into entities. Further mathematical developments and proofs are given in the appendix.

Brooks [1991] accuses the knowledge representation community of spending their efforts on formal anomalies that never arise within in actual programs. The work here certainly looks like that, but I believe that this is an illusion. I do not claim that a physical reasoning program has to worry about clustered variation. I do claim that a physical reasoning program whose creator has not worried about clustered variation is in danger of generating nonsense, and that this danger increases rapidly with the breadth and flexibility of the program. And when situated automata reach a point of sophistication where they take into account more than what is immediately in front of their nose, their creators will also have to confront these issues.

2 Why exclude clustered variation?

The original introduction of the axiom of discrete variation in [McDermott, 82] was motivated by the following example. Suppose we wish to prove the following theorem: "If day is always followed by night, and night is always followed by day, and it is now day, then it will always be either day or night." It is easily seen that this theorem cannot be proven because it is not logically necessary. The following scenario is perfectly consistent: Day from $t=0$ to $t=1/2$; then night from $t=1/2$ to $t=3/4$; then day until $t=7/8$; then night until $t=15/16$ ...and, starting at $t=1$, a state that was neither day nor night. In such a case, day would always be followed by night and night by day, but eventually it would be neither day nor night. (Compare, "If each bounce of the ball is followed by another bounce, then the ball will always be bouncing," which is not necessarily true.) However, if we postulate that "DAY" and "NIGHT" are states with discrete variation, then the proof goes through directly.

A sceptic, however, could argue that this is not a particularly good example to illustrate the usefulness of requiring discrete variation. Suppose that the conclusion, "It is always day or night" does not follow from the premise "Day is followed by night and night by day," what harm is there in that gap? Why, indeed, should one want to derive this by inference at all, considering that the

statement "It is always day or night," is just as reasonable an axiom, with just as much direct evidence, as "Day is followed by night, and night by day?"

Let us therefore modify the example. Consider a finite state system with four states numbered 0, 1, 2, 3; an action A, which adds 2 to the state, mod 4; and an action B, which adds 1 to the state, mod 4 (Figure 3). Now, given that the system is in state 0 in situation S1 and that only action A occurs between S1 and S2, infer that at S2 the system in an even-numbered state. This may be a useful thing to infer. For example, if you are playing a game against an opponent and you wish to keep the game state out of odd-numbered states, it is useful to know that you can do this indefinitely, as long as you can prevent the occurrence of any actions other than A. Another example: if the states are nodes in an envisionment graph from a qualitative physical reasoner, such as QP [Forbus, 85], then the usual interpretation of the graph includes the inference that the system will always be in a state that can be reached on a path through the graph.

The problem is that the result does not follow, if clustered variation is possible. It is consistent with the theory that, if action A occurs once over the first half of the interval, and then a second time over the next quarter, and a third time over the next eighth ... then at time S2, after infinitely many iterations of A, the system should be in either state 1 or state 3. And this is not due to a gap in the logic; this really can happen in systems where clustered variation is possible, such as the bouncing ball. Any single transition "collision with the ground" preserves the state "ball is moving" but the infinite sequence of these transitions does not.[5]

Clearly, we cannot impose an axiom of the form "The system must be in an even-numbered state," analogous to the axiom "It is always DAY or NIGHT;" the system is allowed to be in odd-numbered states at times before S1 or after S2. One approach would be to add a frame axiom, "The system cannot change from the parity of the state between SI and S2 unless an action of type B occurs." But such an approach could easily lead to a horrific version of the frame problem. One can easily construct networks in which each different subset of actions generates a different partitioning of the network into strongly connected components. In such a network, one would need a different frame axioms for each such set of actions; $2^k - 1$ axioms for k different actions. Since all these axioms are computable from the network description, this is preposterously inefficient. The only reasonable approach, therefore, is to demand that the states of the network vary discretely, in which case the conclusion follows directly.

A second example: Suppose we posit the following axioms:

- Given that window W is in good repair in situation $S1$, it will be broken in situation $S2$ if and only if some object has hit it in the interval $(S1, S2]$.
- Object O is responsible for breaking window W if and only if there is a situation S such that W is in good repair up to S and O hits W in S.

We would like to infer the rule that, if W goes from being in good repair to being broken, then some object is responsible for breaking it. Again, this is not a valid inference. Consider the scenario where the window is hit by some object at time $t = 1$; before that, it had been hit at time $t = 1/2$; before that, at time $t = 1/4$; at time $t = 1/8$ ... Then certainly the window is broken at any time $t > 0$. However, since each hit occurs after it is already broken, no object is responsible for breaking it. (This might come up in a question of legal responsibility. Obviously, a person cannot be sued for breaking a window that's already broken.)

[5] This inference may also fail in envisionment graphs derived from theories with quantities of infinitely different orders of magnitude. ([Davis, 89b], p. 429)

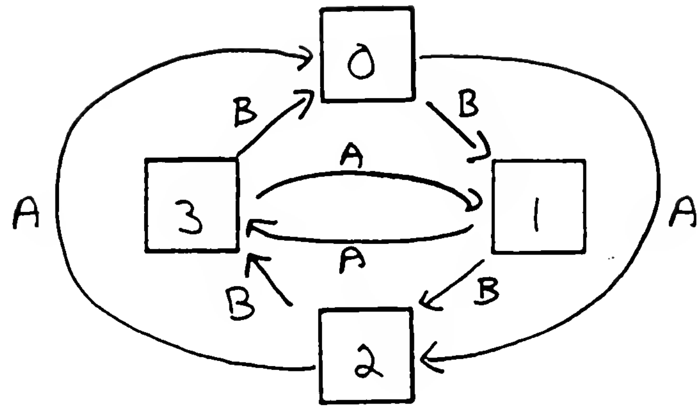


Figure 3: Transition Network

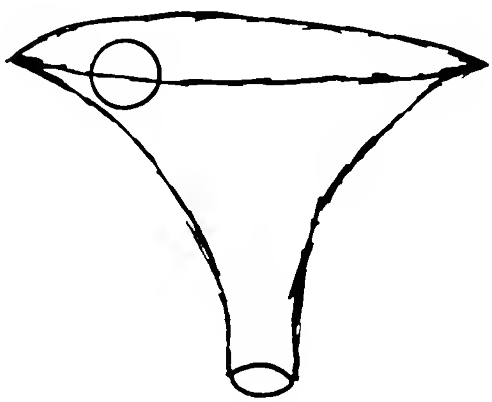


Figure 4: Ball in Funnel

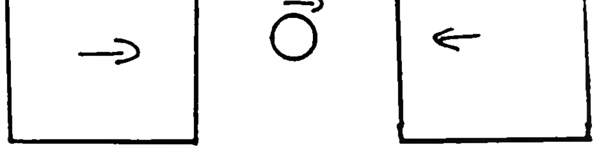


Figure 5: Small object bouncing between two large objects.

3 Other Examples of Clustered Variation

If we adopt standard Physics I idealizations, such as massless strings of zero radius, point masses, and the like, then it is easy to generate scenarios that show clustered variation. However, such models run into all kinds of problems and inconsistencies; clustered variation is the least of their difficulties. Within more reasonable theories of mundane domains, I have found three types of examples of clustered variation: bouncing rigid solid objects, idealized reflected light, and oscillators with a varying control parameter.

Bouncing: Our original example above of clustered variation was a ball bouncing to heights that reduce in geometric progression. Other more complex examples can be contrived: (1) The restoring force need not be gravity; a ball bouncing off a surface, and returned by an electric field or a spring will (in principle) behave the same way. (2) A ball falling through a smoothly curved funnel whose mouth is just as large as the ball and has a vertical tangent will have infinitely many bounces (Figure 4). (3) If the initial velocity and the coefficient of restitution are small enough, a ball bouncing between two heavy objects will bounce infinitely often as the two objects converge to a distance exactly equal to the diameter of the ball (Figure 5). The detailed analysis of (2) and (3) is given in appendix A.

Reflected light: Consider a rectangular box of dimensions $p \times q$ whose internal sides are mirrors. A ray of light is released from one corner of the box with angle θ. The points where successive reflections of the ray will hit the box may be easily calculated by continuing the ray in a straight line, and considering where it intersects virtual reflections of the box (Figure 6). The coordinate c_k where the ray hits the left wall for the kth time can thus be calculated as follows: Let $l_k = 2kp \tan\theta$. If $\lfloor l_k/q \rfloor$ is odd, then $c_k = -l_k \bmod 2q$; else $c_k = l_k \bmod 2q$. However, if $p \tan\theta/q$ is irrational, then no two values c_i and c_j are equal; hence the points c_k are densely scattered along the line.

If we modify the model to use a beam of limited width and uniform intensity and imperfect mirrors that reduce intensity by a fixed fraction, then the whole internal surface will be lit up after finitely many reflections; however, the intensity of illumination will be a fractal function of position. The same will hold if we further generalize the model to allow a limited angular spread and a non-uniform intensity, as long as the intensity cuts off discontinuously at the boundaries of beam.

This clustered variation is spatial rather than temporal, but, as we shall see in section 5.4, the two are closely related. The idealization breaks down, of course, when we consider the finite wavelength of light, the quantum nature of light intensity, or the finite speed of light, but in ordinary applications, it is convenient to idealize away all of these.

Variable frequency oscillation: Most physical oscillators behave, at least around their equilibrium point, like damped harmonic oscillators; that is, they oscillate with fixed period, and exponentially decreasing amplitude. Such a system does not exhibit clustered variation, since its zero-crossings are evenly spaced; in principle, it will cross zero infinitely many times, but only over infinite time. However, if the period depends on an external parameter, and can be made to go to zero by adjusting the parameter, then clustered variation may be achievable. For example, if you run your finger along a vibrating violin string, then the frequency is inversely proportional to the distance from the contact point to the bridge. In principle, if you move your finger toward the bridge in such a way that the distance from your finger to the bridge is proportional to $(t0 - t)^4$ then the string crosses the center infinitely often. (Moving your finger at constant speed actually does not give clustered variation.) Other similar examples abound: a pendulum whose length goes to zero in finite time with the proper time-dependence, an LRC circuit where the capacitance goes to zero, and so on.

The example of the vibrating string is slightly peculiar, in that any single point on the string

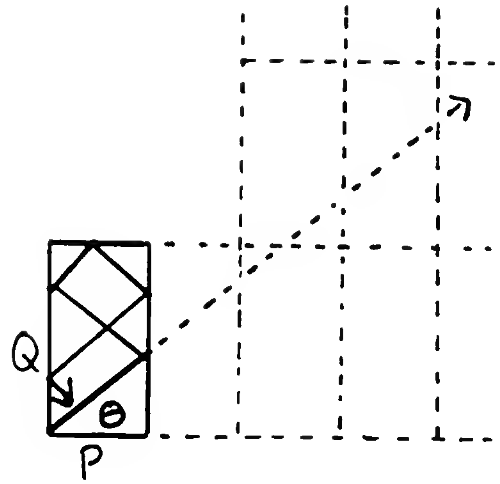
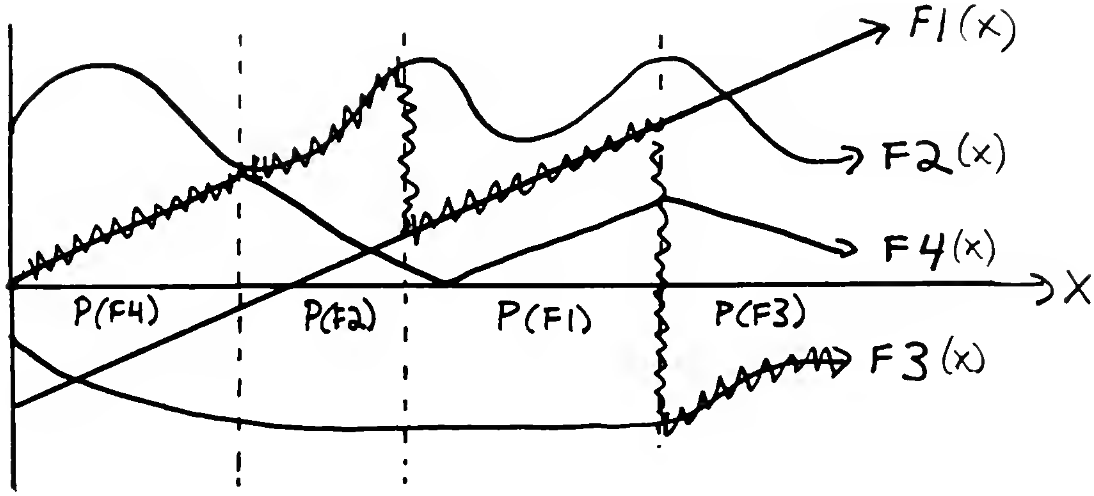


FIGURE 6: Reflections of a light ray



The splice S(x) is marked with hatching

Figure 7: Splicing functions

crosses the center only finitely many times. We will discuss this further in section 7. We can remove this anomaly by supposing that you are moving two fingers together on the string. Then the string at the meeting point crosses the center infinitely often.

4 Infinite sequences of events

In this section, we consider the analogue of clustered variation for events and actions; should it be considered possible for infinitely many events to occur without overlapping during a finite interval?

Certainly, if we admit event types such as "O moves from $P1$ to $P2$," or "O moves distance X in direction D," then Zeno's paradoxes show that the occurrence of infinite sequences of events is unavoidable. The motion of the arrow from 0 to 1 entails the occurrence of the infinite sequence, "Arrow moves from 0 to 1/2; arrow moves from 1/2 to 3/4; ..." Though this looks a little odd, it is presumably not problematic; it is just a clumsy way of speaking about a simple reality. As an analogy, we may observe that, if the sequence of discrete events "sequence(A, B, C, D, E)" occurs then concurrent with occurrence of event C are the occurrences of events such as "sequence(B, C)," "sequence(C, D)", "sequence(A, B, C)" ...The fact that nine such events are occurring simultaneously is not a problem, just a clumsy description.

The problems start to look more troublesome when we switch from descriptive to imperative mode, and talk about plans rather than event descriptions. Do we want to say that Achilles can carry out the plan,

```
procedure P1
while (place(me) ≠ 1) move-to((1 + place(me))/2)
```

starting at place=0 by running forward to 1? If so, what about the plan

```
procedure P2
while true move-to((1 + place(me))/2)
```

or

```
procedure P3
begin D := 1;
      while (place(me) ≠ 1)
            begin D := D/2;
                  go_forward(D)
            end end
```

If such iterative plans are OK, why not recursive plans like

```
procedure go1(D)
begin go_forward(D/2);
      go1(D/2)
end

procedure go2(D)
begin go2(D/2);
      go_forward(D/2)
```

```
end

procedure go3(D)
begin go3(D/3);
      go_forward(D/3);
      go3(D/3)
end
```

Obviously, a plan interpreter implemented along the lines of a conventional programming language interpreter will bomb with each of these; they will either go into an infinite loop that takes infinite time, blow the stack, or run into underflow. However, if plans are viewed as descriptions of actions to be reasoned about rather than as programs to be blindly executed, then there is a good case to be made that these plans should be admitted just as we admitted the event descriptions discussed above.

It is, in fact, quite easy to define the semantics of planning languages in a natural way that admits these plans as correct. For example, most semantics for plans with conditionals and loops (e.g. [McDermott, 82] [Manna and Waldinger, 86]) make the assumption that the evaluation of the test is instantaneous, and that the body of the conditional begins simultaneously with the conditional. If so, and we idealize real arithmetic as perfect, then P1 is valid. The validity of P2, under the same assumption, depends delicately on exactly how the termination conditions for a loop are defined. The assignment statements in P3 need a little more care. Clearly, the execution of an assignment statement cannot be instantaneous, since that would leave the order of execution of a sequence of assignments badly undefined. However, if no minimum cycle time is set for the execution of an assignment — for example, if it is consistent with the semantics that the assignment should take a nanosecond on the first iteration, half a nanosecond on the second, a quarter on the third, and so on — then there are executions of P2 that terminate in finite time. As for the recursive programs go1, go2, go3, by using an appropriate minimal fixed-point definition of recursion, we can make these valid as well. (In effect, we define the execution of any infinitely deep calling sequence as completed instantaneously.)

Why should we care what the semantics does, given that no plan interpreter can execute these plans, and no planner will ever come up with them? Actually, these assumptions may not be true. At the interpretive side, it is not hard to imagine a plan optimizer that could recognize one or another of these forms and compile it into the instruction to move forward. (Compare "tail-recursion" optimization, which allows an infinite loop for an interactive program to be written recursively, without ever blowing the stack.) At the planning side, it is not as obvious as it looks at first glance that no planner will ever come up with these. In fact, if you consider Achilles' problem of overtaking the tortoise and apply a GPS-like stategy, then the infinite loop

while (place(me) $\neq$ place(tortoise)) move-to(place(tortoise))

is exactly what you generate. In fact, if the motion of the tortoise is not known in advance and cannot be extrapolated, all plans for overtaking the tortoise will involve something like this. Achilles can overtake a slowly moving butterfly, but he must be prepared to make finer and finer adjustments as he gets closer and closer. (It should be noted that his motion will not, in general, observe the condition of piecewise analyticity discussed in section 5.) Of course, this is only a truly infinite loop if one assumes infinitely precise perception and motion, but these are natural idealizations until one gets down to very nitty-gritty levels of robotic detail.

Either way we go on this choice makes substantial trouble in reasoning about the plans. If we rule out infinite loops by requiring (realistically) that entering each control structure requires a cycle of time, then this complicates plan verification because it becomes necessary to demonstrate that

the world does not change too much while the control structures are being interpreted. If we admit infinite loops, then we have to reason about the effect of an infinite loop, which gets back to the problems mentioned in section 2. Moreover, rules such as "A recursive program with no base case is invalid," is very useful, and it would be a pity to give it up.

5 Consequences of discrete variation for Fluents

In this section, we consider how requiring discrete variation affects the range of states and fluents that can be considered. We consider Boolean combinations of states (section 5.1); fluents onto finite ranges (5.2); real-valued fluents (5.3); and spatial fluents (5.4). We also show that guaranteeing discrete variation also often involves restricting the local area, or the universe as a whole, to contain only finitely many objects.

5.1 Boolean combinations of states

It is easily shown that, if $A1$ and $A2$ both vary discretely then the conjunction of $A1$ and $A2$, the disjunction of $A1$ and $A2$ and the negation of $A1$ all vary discretely. Thus, any Boolean combination of finitely many such states varies discretely. For future reference, we give this fact a number:

Theorem 1: If $A1 \dots Ak$ each has discrete variation, then any Boolean combination of them also has discrete variation.

The Boolean combination of infinitely many states may have clustered variation even if each single state varies discretely.

5.2 Fluents with Finite Range

Let F be a fluent with a finite range D. The state $F(t) = c$ varies discretely for every constant $c \in D$ if and only if F changes values only finitely many times in each finite interval.

Let $F1 \dots Fk$ be fluents with finite range and discrete variation; let O be a k-place function, and let P be a k-place relation. Then the fluent $\lambda(t)O(F1(t) \dots Fk(t))$ and the state $\lambda(t)P(F1(t) \dots Fk(t))$ varies discretely. Recursively, any fluent or state define as a ground term over $F1 \dots Fk$ varies discretely.

5.3 Real-Valued Fluents

When we move to real-valued fluents, things get trickier. The requirement that $F(t)$ change its value only finitely many times in a finite interval is much too strong; it would rule out any function that changed continuously, such as the identity function $F(t) = t$. On the other hand, the requirement that the state $F(t) = c$ have discrete variation for each constant c is much too weak. The bizarre function

$$F(t) = \begin{cases} t & \text{if } t \text{ is rational} \\ -t & \text{if } t \text{ is irrational} \end{cases}$$

has discrete variation for each state $F(t) = c$, but has clustered variation for the states $F(t) > 0$ and $F(t) < 0$.

We can rule out such weirdnesses by requiring that both states $F(t) = c$, and $F(t) > c$ have discrete variation. But even this stronger condition does not allow us to do everything we want to.

For example, the function $F(t) = t$ and $G(t) = t + t^3 \sin(1/t)$ both satisfy this condition. (Note that $G(t)$ is monotonically increasing in a neighborhood of t=0.) However, the state $G(t) > F(t)$ has clustered variation in the neighborhood of $t = 0$. We would like to be able to perform basic operations on our fluents without worrying about getting into trouble.

Luckily, there is a helpful theorem about power series. First, three definitions:

Definition 2: A non-empty subset I of the real line is an *interval* iff it satisfies the following conditions: if $X \in I$, $Y \in I$, and $X < Z < Y$ then $Z \in I$. This includes open, closed, half-open, and single-point intervals, bounded or unbounded.

Definition 3: A function $f(x)$ on the reals is *analytic* at point x_0 if

i. $f(x)$ is continuous and infinitely differentiable at x_0;

ii. The Taylor series

$$f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \dots$$

converges to $f(x)$ for all x in some neighborhood of x_0.

(An equivalent definition is that $f(x)$ is extendable to a differentiable function over a neighborhood x_0 in the complex plane. Condition (i) alone, that $f(x)$ is continuous and infinitely differentiable, is not sufficient to guarantee discrete variation, contrary to a misstatement in [McDermott, 82]. An example is the function $f(x) = e^{-1/x^2} \sin(1/x)$ for $x \neq 0$. $f(0) = 0$.)

Definition 4: A function f is analytic over an interval I if f is analytic at every point $x \in I$.

We now have the following useful theorems: (The proofs of these may be found in any standard text on infinite series or complex variables.)

Theorem 5: If F is analytic, then the states $F(t) = c$ and $F(t) > c$ vary discretely.

Theorem 6: If f and g are analytic, then the functions $f+g$, $f-g$, fg and $f(g)$ are all analytic. The integral and derivative of f are analytic. The quotient $f(x)/g(x)$ is analytic except for x where $g(x) = 0$. The power $f^{\alpha}(x)$ is analytic for all x if α is a non-negative integer, and analytic for all x where $f(x) > 0$ for all other α. The inverse of f, restricted to a range where it is uniquely defined, is analytic at any point where $f'(x) \neq 0$.

Remark 7: A lot of good functions are analytic for all x, including polynomials, exponentials, sine and cosine, and so on. The logarithm is analytic for all positive x. Most real-valued functions that come up in practice are analytic for almost all x.

One type of closure that cannot be achieved is closure under solving differential equations. There are differential equations that do not superficially look badly behaved but whose solutions include functions with clustered variations. An example is the initial-value system.

$\dot{y} + y^2 = 0.$
$\ddot{x} + y^4 x = 0.$
$y(-1/\pi) = -\pi;\ x(-1/\pi) = 0;\ \dot{x}(-1/\pi) = -\pi.$

For $t < 0$, this has the solution $y(t) = 1/t$; $x(t) = t \sin(1/t)$.

The class of analytic functions, though large, is not adequate for AI applications, which need to model discrete functions. Often, the behavior of a fluent will change suddenly, even discontinuously, as the result of an event such as a switch flipping, a moving object colliding with an obstacle, a heating pot of water reaching boiling point, and so on. If we can suppose that such event occur only

at discrete times, then we can widen the class of functions to include the corresponding fluents as follows:

Definition 8: Let $f_1, f_2 \ldots$ be a collection (finite or infinite) of real-valued functions. Let $P(f)$ be a function that associates a real interval with each f_i. Assume that P has the following two properties:

i. P is disjoint and exhaustive. That is, for any real x there is a unique f_k such that $x \in P(f_k)$.

ii. P varies discretely. That is, if J is a bounded interval, then J contains $P(f_i)$ for only finitely many i.

We then define the *splice* of P as the function $s(x)$ such that $s(x) = f_i(x)$ for all $x \in P(f_i)$. (Figure 7)

Note that P may have infinitely many different values, as long as it varies discretely.

Definition 9: A function $s(x)$ is *piecewise analytic* if there is exists a collection $f_1, f_2, \ldots$ and a function $P(f_i)$ such that

i. P satisfies the conditions of definition 8 above.

ii. $s(x)$ is the splice of P.

iii. Each f_i is analytic over an open set that contains the closure of $P(f_i)$.

Note: In condition (iii) above, it is important that f_i be analytic over the closure of $P(f_i)$, not just over $P(f_i)$. For example, the function $f(x) = x \sin(1/x)$ is analytic over the open interval $(0, \infty)$, but not at 0. Hence, the function defined to be 0 for all $x \leq 0$ and $x \sin(1/x)$ for all $x > 0$ is not considered piecewise analytic.

We then have the following theorems:

Theorem 10: If f is piecewise analytic, then the states $f(x) = c$ and $f(x) > c$ vary discretely.

Theorem 11: The class of piecewise analytic functions satisfies all the closure properties enumerated for analytic functions in theorem 6 (assuming that some convention is adopted for defining the derivative at points where the function is not differentiable.) Also, if $f(x)$ and $g(x)$ are piecewise analytic, then $\max(f(x), g(x))$ and $\min(f(x), g(x))$ are piecewise analytic.

The proofs are immediate from theorems 5 and 6, together with definitions 8 and 9.

There are some functions that might be useful that still not included in this definition, including divergent functions, like $1/x$ or $\sec(x)$, and fractional powers, such as $\sqrt{x}$. If we need these, we can define a class of piecewise Dirichlet functions. The definition of these is rather complex and is deferred to Appendix A, section 8.3. Here, we adduce the following results:

- If $f(x)$ is piecewise Dirichlet then the states $f(x) = c$ and $f(x) > c$ vary discretely.
- All analytic functions are piecewise Dirichlet.
- The class of piecewise Dirichlet functions is closed under addition, subtraction, multiplication, non-zero division, and differentiation, assuming a convention for defining the derivative at non-differentiable points. If $f(x)$ is piecewise Dirichlet in $\mathcal{F}$ and is positive over interval I, then $f^{\alpha}(x)$ is piecewise Dirichlet for any real α.

However, we lose closure under integration and composition.

By theorem 1, if the states $f(x) = c$ and $f(x) > c$ vary discretely, then any finite Boolean combination of these also varies discretely. But a finite Boolean combination of inequalities is just a finite union of intervals. Thus, if S is the union of finitely many intervals, then the state $f(x) \in S$ varies discretely if $f(x)$ is piecewise analytic or piecewise Dirichlet.

What about more general states involving the value of $f(x)$? Clearly, if a state $P(y)$ has clustered variation as a function of y, then it would be unreasonable to expect $P(f(t))$ to vary discretely as a function of t; even such a well-behaved function as $f(t) = t$, would violate that condition. For example, we cannot expect the state, "$f(t)$ is a rational number" to vary discretely. However, it is easily shown that any set of real numbers that does not have clustered variation is the union of a discretely varying collection of intervals. We have already taken care of the case of a finite collection of intervals; the only remaining case, therefore, is an infinite collection of discretely varying intervals, such as "$f(t)$ is an integer," or "$\sin(f(t)) > 0$.

If I is the union of infinitely many intervals with discrete variation, and $f(x)$ is piecewise Dirichlet and bounded in every finite interval, then the state $f(x) \in I$ varies discretely. If $f(x)$ diverges for finite x, then, in general, the state $f(x) \in I$ does not vary discretely. Thus, a state such as "$f(t)$ is an integer" or "$\sin(f(t)) > 0$" has discrete variation for functions such as $f(t) = t$ or $f(t) = e^t$, but not for $f(t) = 1/t$.

We run into problems, though, if we have two scales, one of which is the reciprocal of the other. In that case, an infinite sequence of discrete intervals in the one corresponds to a cluster point in the other. For example, consider the relation on tones, "This tone is a C." If we consider the series of higher and higher C-tones in octaves, the frequencies form a discrete series of numbers, but the wavelengths are clustered around zero; if we consider lower and lower C-tones, the wavelengths are discrete but the frequencies are clustered around zero.[6]

It is true that both are discretely spaced on a logarithmic scale. However, a logarithmic scale can only be used if that zero and negative values are guaranteed to be meaningless. We can get away with this if we are using the scale only for wavelengths, but if we wish to measure wavelength using the same scale as for other quantities of the same dimensionality such as distances or coordinates, the assumption is not acceptable.

Of course, whether the state $P(f(t))$ has discrete or clustered variation over time does not depend on of the scale used to measure the range space of f. (For example, whether the state "The string is vibrating at the tone of C" varies discretely over a time period is a physical fact, which does not depend on whether we define C in terms of frequency or wavelength.) The point is that it is not sufficient to require that the real-valued fluents be well-behaved; it is also necessary to show that the significant states in their range spaces vary discretely; and this latter assumption is not one that can be made universally about every quantity space.

We can show the above closure results to analyze relations among multiple fluents. For example, if $f(x)$ and $g(x)$ are piecewise analytic, then so is $f(x) - g(x)$, by the closure theorem 11. Therefore, the relation $f(x) > g(x)$, which is equivalent to $f(x) - g(x) > 0$, must vary discretely.

5.4 Spatial Fluents

Reasoning about shape and motion raises problems that are analogous to those of real-valued fluents, but in a more complex setting. We wish to guarantee that the times in which a moving point object are inside a specific region vary discretely; that the times in which two moving regions overlap, or abut, or are within a given distance of one another vary discretely; and so on. Ensuring these

[6] Going up an octave doubles the frequency, and halves the wavelength.

conditions requires restricting both possible shapes and possible motions.

As in the one-dimensional case, we wish both to include both a wide range of possible shapes and motions, and to attain useful closure conditions. However, I have not been able to prove discrete variation, even over the set of shapes and motions generated by simple kinematic theories. Suppose, that we wish to posit that all physically real motions and shapes are drawn from a set of "acceptable" motions and shapes. Presumably, this set will have to satisfy certain closure conditions. It would not be acceptable, for example, to propose a theory in which a machine that could be built out of acceptable shapes would generate a motion that was not an acceptable motion. A modest set of closure conditions might include the following:

- Uniform translation and uniform rotation are acceptable motions.
- The class of acceptable motions is closed under composition. That is, if each of two coordinate systems is undergoing an acceptable motion relative to a third, then their motion relative to each other is acceptable.
- A sphere is an acceptable shape.
- The connected components of the normalization of the union, intersection, or set difference of two acceptable shapes are acceptable.
- If an acceptable shape undergoes an acceptable motion for a finite time, the shape that is swept out is acceptable.
- If a collection of acceptable shapes are assembled into a linkage with one degree of freedom, and one point in the linkage has an acceptable motion, then all points in the linkage have acceptable motions.

Unfortunately, I have not been able to show that the class of shapes and motions generated by these closure conditions satisfies the discrete variation condition. The strongest result I have show is that the class of semi-algebraic shapes with piecewise analytic motions satisfies the discrete variation condition, but this class is not closed under "sweeping out". If the motions are restricted to be algebraic, then the closure conditions hold, but they no longer include uniform rotation, which is trigonometric. If the shapes are generalized to be analytic, then I have not been able to prove discrete variation or closure conditions, though I conjecture that both hold.

Some partial results are stated below.

Definition 12: A *region* is a subset of $\Re^k$.

Definition 13: An *algebraic constraint* is a constraint of the form $f(\vec{x}) > 0$ or $f(\vec{x}) \geq 0$ where f is an algebraic (multinomial) function. A *semi-algebraic* region is the solution space of a finite Boolean combination of algebraic constraints.

Theorem 14: Let $\mathbf{R}$ be a semi-algebraic region, and let $\vec{x}(t)$ be a piecewise analytic function of time. Then the Boolean fluent $\vec{x}(t) \in \mathbf{R}$ varies discretely.

Proof: Let $\mathbf{R}$ be defined by the Boolean combination of the functions $f_1, f_2 \ldots f_k$. The functions $f_i(\vec{x}(t))$ are piecewise analytic functions in t; hence each constraint $f_i(\vec{x}(t) \geq 0$ varies discretely, by theorem 5. Hence, by theorem 1, the Boolean combinations of these states also varies discretely.□

Theorem 15: (Informal statement) Consider two objects that are moving rigidly through space. If the shapes of the objects are semi-algebraic, and the motions are piecewise analytic, then states such as "The two objects overlap," "The objects abut," or "The distance between the objects is less than D" vary discretely. This result can be extended to objects that move non-rigidly, as long as the shape transformations are characterized by algebraic functions.

Appendix A contains a precise statement and proof of theorem 15.

Definition 16: An *analytic constraint* is a constraint of the form $f(\vec{x}) > 0$ or $f(\vec{x}) \geq 0$ where f is an analytic function. A *semi-analytic* region is the solution space of a finite Boolean combination of analytic constraints.

Theorem 17: Let $\mathbf{R}$ be a semi-analytic region, and let $\vec{x}(t)$ be a piecewise analytic function of time. Then the Boolean fluent $\vec{x}(t) \in \mathbf{R}$ varies discretely.

Proof: Identical to proof of theorem 14.

Conjecture 18: Consider two objects that are moving rigidly through space. If the shapes of the objects are semi-analytic and the motions are piecewise analytic, then states such as "The two objects overlap," "The objects abut," or "The distance between the objects is less than D" vary discretely.

Another issue concerns real-valued functions defined over space, such as temperature or pressure. We would like to guarantee that the values of such functions encountered by an object moving through space are well-behaved. We can achieve this with a generalization of the notion of a piecewise analytic function:

Definition 19: Let $s(\vec{x})$ and $f_1(\vec{x}), f_2(\vec{x})\ldots$ be functions from $\Re^k$ to $\Re$. Let P be a function mapping each of the functions f_i to a region in $\Re^k$ such that

a. P is disjoint and exhaustive. That is, for each $\vec{x} \in \Re$ there is exactly one f_i such that $\vec{x} \in P(f_i)$.

b. For all i, $P(f_i)$ is a semi-analytic region of $\Re^k$.

c. Any bounded $S \subset \Re^k$ interesects $P(f_i)$ for only finitely many i.

d. f_i is analytic over an open set that includes the closure of $P(f_i)$.

e. For $\vec{x} \in P(f_i)$, $s(\vec{x}) = f_i(\vec{x})$.

Then $s(x)$ is said to be *piecewise analytic.*

Theorem 20: Let $\vec{x}(t)$ be a piecewise analytic function from time to R^k, and let $s(\vec{x})$ be a piecewise analytic function from $\Re^k$ to $\Re$. Then $s(\vec{x}(t))$ is a piecewise analytic function of time.

Proof: Immediate from the definition and theorem 17.□

5.5 Finitely Many Objects

If the universe contains infinitely many objects, then states of the form "There exists an object with property P" may have clustered variation even if each individual object is well-behaved. For example, suppose there is an infinite collection of light bulbs, and bulb #1 is on from $t = 0$ to $t = 1/2$, bulb #2 is on from $t = 3/4$ to $t = 7/8$; bulb #3 is on from $t = 15/16$ to $t = 31/32$ and so on. Then the state "Some light bulb is on" has clustered variation, even though each individual light bulb is well-behaved.

Often, states of physical significance depend only on objects that are in some bounded region. In that case, it suffices to demand that in there are only finitely many objects in any bounded region of space-time. Suppose, modifying the above example, we have a rule that a room is illuminated just if it contains a lighted light bulb. In that case, it is possible to guarantee that the state "The room is illuminated" varies discretely by positing that the state "Light bulb N is on" varies discretely for each N, and that during any finite time interval there are only finitely many light bulbs in the room.

Note that this condition is strictly stronger than the condition, "At any instant of time, there are only finitely many objects in any spatially bounded region," which does not suffice to guarantee the desired result. Consider the following example. Syldavia is interested in the state "There are no enemy aircraft in Syldavian airspace." Freedonia has an infinite fleet of airplanes, regularly spaced on an infinite landing strip. At time $t = 0$, plane 1 flies from its spot on the landing strip to Syldavia, returning to its spot by $t = 1/2$. Plane 2 does the same from $t = 3/4$ to $t = 7/8$, and so on. No individual plane is doing anything extraordinary (except for going faster than the speed of light), and the planes can all be of a standard shape and size; there is no need to posit that the plane size is gradually decreasing, or anything of the kind. Nonetheless, the state "There are no aircraft in Syldavian airspace" has clustered variation.

Imposing the condition of only finitely many objects in a bounded region of space-time may, in its turn, require further restrictions on processes that create objects or that send objects into regions. For instance, if we replace the airplanes of the previous example by bullets fired from guns spread out on the landing strip, with further guns firing faster and faster bullets, then we must posit that the guns do not fire in such a way as to send infinitely many bullets through a finite space in finite time. This, note, is a restriction on processes that happen over an infinite spatial region. If we space the guns and fix the velocities carefully enough, we can even work it so that the further guns must fire earlier and earlier; in that case, it is a restriction on processes that occur over an unbounded region in both space and time. (If gun I is at $x = 2^{4I}$ and fires a bullet in the negative direction at time $-2^{2I} - 1/2^{2I-1}$ at speed 2^{2I}, then the bullet will be between $x = 0$ and $x = -1$ between the times $-1/2^{2I-1}$ and $-1/2^{2I}$.) Of course, the guns here are superfluous; one can imagine that the bullets have just been pursuing these trajectories since time immemorial, never looking in the least clustered until $t = 0$. All in all, it may simply be best to posit that in all space-time there are only finitely many objects.

An example of creating objects: suppose that the bouncing ball of section 1 is slightly brittle, and that each time the ball hits the ground a small piece breaks off whose size is proportional to the energy of the collision. Then, after the bouncing ball has attained a state of rest, the floor will be littered with an infinite collection of diminishing objects, which can proceed to make trouble.

It is often useful to require that only finitely objects exist in an bounded spatial region for reasons independent of temporal considerations. For instance, in the blocks world, one would like to infer that, if a block is on top of a tower of blocks, then it is resting on some particular other block; this is guaranteed if there are only finitely many blocks in the tower. Similarly, in a theory of perception, one would like to infer that, if there is an object blocking your view in a particular direction,. then there is a frontmost object in that direction; again, this is guaranteed if there are only finitely many objects in the neighborhood, and objects are reasonably shaped. (Figure 8)

6 Ontology

The status of discrete and clustered variation depends critically on the way in which the universe is divided into entities.

One example was alluded to in section 3. Suppose I have a vibrating violin string, and I run my finger to the bridge in a finite time in such a way that the part of the string between the finger and the bridge vibrates infinitely often. If we consider "The vibrating part of the string" to be an entity, then the state "The vibrating part of the string is to the left of center" has clustered variation. However, if we take the individual points on the string as primitive entities, then these are all well behaved. Each point crosses the center line a finite number of times, and then, being on the far side of the finger, remains still.

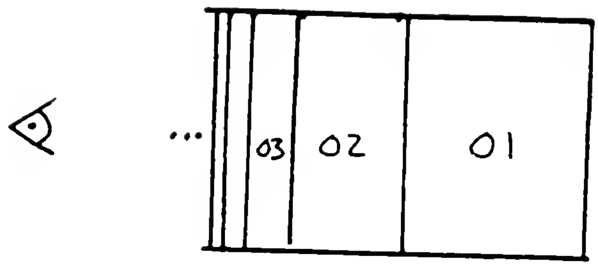


Figure 8: Infinite collection of occluding objects

In a recent paper axiomatizing a theory of cutting solid objects [Davis, 91a], I ran into this issue in an extreme form. In this paper, I proposed an idealized model of cutting in which the blade change the shape of the target by simply annihilating the material that it swept through. There were two natural approaches to axiomatizing this theory. The more obvious approach is to view the target as an object whose shape is gradually modified by the blade until it is split, at which time the original object ceases to exists and two new objects come into being. In this theory, there are three central axioms of change:

1. If an object exists from $S1$ to $S2$, then its shape in $S2$ is equal to its shape in $S1$ minus whatever has been carved out of it.

2. Object O ceases to exist in situation S if and only if its shape is connected up until S and becomes disconnected in S.

3. Object O comes into existence in situation S if and only if it is one connected component of another object $O2$ that ceases to exist in S.

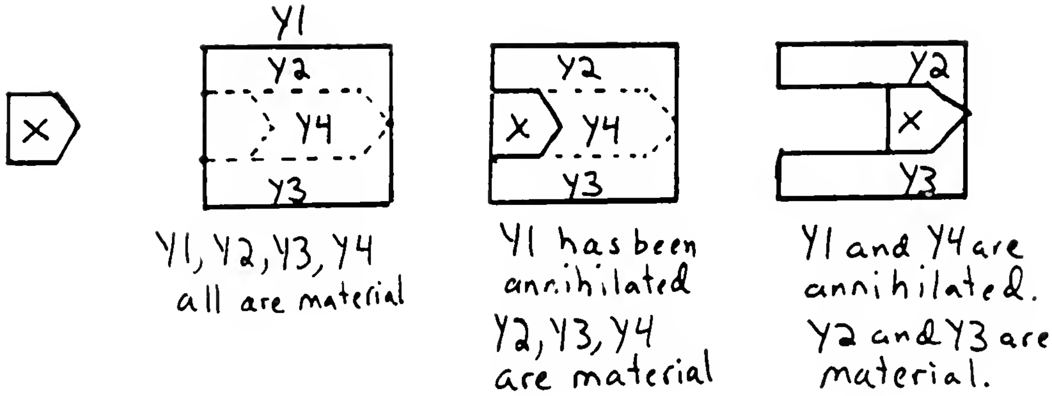
These axioms serve both as causal axioms and as frame axioms. The first axiom is causal if something has been carved out of the object, and is a frame axiom if nothing has. The second and third axioms are causal in the direction "Change if condition" and frame axioms in the direction "Change only if condition."

The more subtle approach takes a chunk of stuff as its primary type of entity. There is one chunk for every (reasonably shaped) subset of the material of an object; thus, there are uncountably many chunks at any instant. A chunk has a fixed shape. It ceases to exist when any part of its material is penetrated by a blade. The one rule of change in this theory is that a chunk is annihilated if and only if it is penetrated by a blade. (Figure 9).

The theory of objects is terribly sensitive to clustered variation. Since dramatic discrete changes occur whenever an object is split, the prospect of infinitely splits occurring in finite time sends the whole theory into conniptions. In a situation like figure 10.A, where, intuitively, the blade cuts an infinite sequence of small objects on the right off the large block at the left, the theory absolutely refuses to admit that the block on the right can still exist when the blade has reached the bottom. An object, in this theory, can come into existence only be being split off a previous object; and here there is no previous objects, only a blur of previous objects. (Keep in mind that the object changes identity each time a split is complete.) Even more peculiarly, a blur of objects can appear out of thin air if the splitting of small pieces is clustered into the past (Figure 10.B) It is therefore necessary, in this theory, to enforce rigidly the condition of discrete variation, which, in turn, means that motions and shapes must be well behaved, and that only finitely many objects are allowed in any finite piece of space-time.

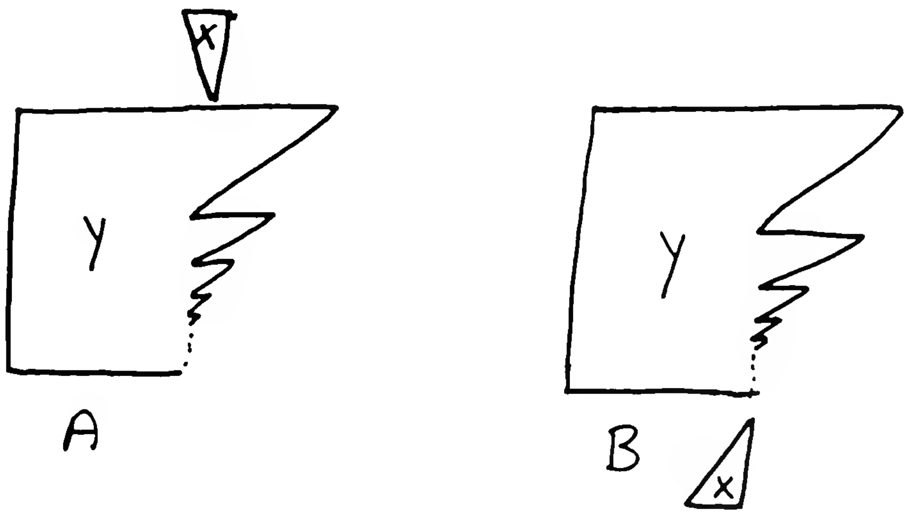
By contrast, none of these anomalies give the slightest trouble to chunk theory. In chunk theory, splitting looks exactly the same as shape deformation; history consists purely of a stream of chunks being destroyed. No restrictions on clustered variation or shape or motion or number of objects need be enforced at all. In fact, since "object" is not a first-order concept in the language of chunks, there is no way even to state these restrictions.

The two theories are provably equivalent in cases that obey the restriction that any bounded region of space-time contains only finitely many objects.



As blade X cuts through target Y, it first annihilates chuck Y1 ago soon as the blade penetrates. Later, it annihilates Y2 as well.

Figure 9: Cutting in terms of chunks



The piece of Y on the left changes its identity infinitely many times as X cuts through.

Figure 10: Anomalies in the theory of objects

7 Conclusion

Perhaps the whole issue still seems spurious and irritating. However, if we follow the conventional practice in knowledge representation of characterizing theories rigorously, and the conventional practice in physical reasoning of using real-valued time and space and using simple physical approximations, then these problems are unavoidable. Those researchers who have tried to develop formal theories of physical reasoning based on real-valued quantities will, I think, agree that these infinitary paradoxes are not a side issue, of interest only to those with a prurient taste for paradox. Rather, developing such theories is like designing structures for the area around the San Andreas Fault. Making a structure earthquake-proof is an irritating distraction from the main task of building a useful structure, but it is really not safe to ignore the issue. Nor is there a simple formula that can be uniformly applied; there are some general techniques, but each new type of structure must be thought through carefully. Some, of course, would argue that this is a sign that one should build in some other region. But no region is perfect, and the Bay area has many advantages: the climate, the lifestyle ...and it's where the action is happening.

8 Appendix A

8.1 Definition of Clustered Variation

In this section, we present two formal definitions of clustered variation and prove them equivalent.

Definition 21: An interval I is a *zone* of boolean fluent A if A has the same truth value throughout I. I is a *maximal zone* of A if I is a zone of A and no interval proper containing I is a zone of A.

Lemma 22: For every situation S, there is a unique maximal zone of A containing S.

Proof: S is certainly contained in at least one zone of A, namely the point interval $[S, S]$. Let I be the union of all zones of A containing S. It is straightforward to verify that I is a zone of A, and that every other zone of A containing S is a subset of I.□

Lemma 23: If $S1 < S2 < S3$, and the truth value of A at $S2$ is different from its truth values at $S1$ or $S3$, then there is no zone containing both $S1$ and $S3$. In particular, $S1$ and $S3$ are in different maximal zones.

Proof: Immediate from the definition of zone.□

Lemma 24: If A has the same truth value at $S1$ and $S2$, and $S1$ and $S2$ are in different maximal zones, then there is an $S3$ such that $S1 < S3 < S2$ and A has the opposite truth value at $S3$.

Proof: If A has the same truth value at all points between $S1$ and $S2$ as it does at $S1$ and $S2$, then the interval $[S1, S2]$ is a zone, so $S1$ and $S2$ would be in the same maximal zone.□

Definition 25.A: A Boolean fluent F has *discrete variation* if any finite interval I contains only finitely many maximal zones. Otherwise it has *clustered variation*.

The above definition intuitively corresponds to the desired meaning of clustered variation. However, it is phrased in high-order terms, posed in terms of the cardinality of a set of intervals, each interval being a set of situations. This may make it difficult for the human reasoner to use, and even more difficult for the automated reasoner.

An alternative equivalent definition, expressible purely in terms of situations, can be formulated:

Definition 25.B Fluent A has discrete variation if, for each situation S, A has a constant truth

value throughout some open interval $(S1, S)$ ending in S and a constant truth value throughout some open interval $(S, S2)$ starting in S. Note that the condition "A has a constant truth value throughout the interval (SP, SQ)" can be expressed purely in terms of situations.

$$\text{constant}(A, SP, SQ) \Leftrightarrow$$
$$[SP < SQ \wedge \forall_{SX,SY}\ SP < SX < SY < SQ \Rightarrow [\text{holds}(SX, A) \Leftrightarrow \text{holds}(SY, A)]$$
$$\text{discrete}(A) \Leftrightarrow$$
$$[\forall_S \exists_{S1,S2}\ \text{constant}(A, S1, S) \wedge \text{constant}(A, S, S2)]$$

We will want the following lemma in proving the two definitions equivalent.

Lemma 26: Let $\mathcal{T}$ be a infinite totally ordered set. Then it is possible to construct either an infinite increasing sequence $S_1 < S_2 < S_3 \ldots$ or an infinite decreasing sequence $S_1 > S_2 > S_3 \ldots$ where each $S_i \in \mathcal{T}$.

Proof: One of two things must be true of $\mathcal{T}$

a. There is a non-empty subset $\mathcal{T}_0$ of $\mathcal{T}$ that has no largest element.

b. Every non-empty subset of $\mathcal{T}$ has a largest element.

In case (a), let S_1 be any element of $\mathcal{T}_0$; let S_2 be any element of $\mathcal{T}_0$ that is larger than S_1 ... This is an infinite increasing sequence.

In case (b), let S_0 be the largest element of $\mathcal{T}$ and let $\mathcal{T}_1 = \mathcal{T} - \{S_0\}$. Let S_1 be the largest element of $\mathcal{T}_1$, and let $\mathcal{T}_2 = \mathcal{T}_1 - \{S_1\}$. In general, let S_i be the largest element of $\mathcal{T}_i$ and let $\mathcal{T}_{i+1} = \mathcal{T}_i - \{S_i\}$. Then the sequence $S_0, S_1, S_2 \ldots$ is an infinite decreasing sequence.□

Theorem 27: Definitions 25.A and 25.B are equivalent.

Proof: Part 1: Suppose that fluent Q is clustered by definition B. Let S be a situation where the discreteness condition does not hold. Assume without loss of generality that there is no interval below S where Q is constant. Then any open interval ending in S contains both situations in which Q is true and situations in which Q is false. Let S_1 be a situation before S in which Q is true. Let S_2 be a situation in (S_1, S) where Q is false. Let S_3 be a situation in (S_2, S) where Q is true. In general, let S_i be a situation in (S_{i-1}, S) where Q has the opposite truth value than it had in S_{i-1}. By lemma 23, each of the situations $S1, S2, \ldots$ must lie in a different maximal zone. Hence there are infinitely many maximal zones in the finite interval $(S1, S)$. So Q is clustered by definition A.

Part 2: Suppose that fluent Q is clustered by definition A. Let $[S1, S2]$ be a finite interval containing infinitely many maximal zones. From each maximal zone Z, choose a point $S_Z \in Z$. By lemma 26, we can either choose an infinite increasing sequence or an infinite decreasing sequence from the S_Z's; assume, without loss of generality, that we have an increasing sequence $S_1 < S_2 < S_3 \ldots$, where no two of the S_i's are from the same zone. By lemma 24, if Q has the same truth value in S_i and S_{i+1}, we can find a situation T_i between S_i and S_{i+1} where Q has the opposite truth value. Interpolating these T_i where necessary, we get a increasing sequence of points with alternating truth values of Q. Since all these points are less than $S2$, the sequence must converge to some value S. Thus, any open interval bounded above by S must contain both points where Q is true and points where Q is false, so Q is clustered by definition B.□

8.2 Analysis of the examples of clustered physical behavior in section 3

Example 2: Let B be a spherical ball of radius r_0. Let F be a funnel whose inner surface is smooth

and radially symmetric about the z-axis, such that the minimum radius of a horizontal cross section is r_0. Let us call this cross-section of minimal radius the "mouth" of F. Suppose that B is dropped without spinning onto the surface of F. We wish to show that B will reach the mouth of F in finite time after bouncing infinitely many times.

Proof: Since the radius at the mouth is minimal, it follows that F is tangent to a vertical cylinder along its mouth. Therefore, if B is inside the mouth of F at time t, its velocity at t must be directly vertical. However, an object moving on a parabolic curve is never moving vertically, unless it is dropped from directly above, and an object rolling or sliding on a curved surface will always "fly off' before the surface becomes vertical. Therefore, the ball cannot reach the mouth, either in any single bounce that begins on the surface of F, or in rolling along F. Therefore, it can only reach the mouth after an infinite collection of bounces on F.

We can show that it eventually does reach the mouth, by showing that there is no way it can indefinitely support itself above the mouth. There is no position of static equilibrium of the ball in the funnel. If we assume that objects can roll perfectly without loss of energy, then there might be a state of dynamic equilibrium where the ball is rolling around the center of the funnel and using the centrifugal force to keep itself up. However, since the starting situation has reflective symmetry in the plane that contains the center of the ball and the vertical axis of the funnel, all subsequent situations must have the same symmetry. In particular, the ball cannot develop a velocity around the axis of the funnel. No other possibilities remain.□

Example 3: Consider an object of mass M moving to the right, and a ball of mass m bouncing between the object and a fixed wall. Suppose that the collisions between the wall and the ball and between the object and the ball are partially elastic with coefficient of restitution μ. For simplicity, we assume that both object and ball move without friction, so that the only loss of energy is due to the collisions between the ball and the object. However, the same results hold if the ball is bouncing between two moving large objects, and if there is sufficiently small loss to friction.

Let v_i be the velocity of the object before the $i+1$st collision between object and ball; z_i be the velocity of the ball after the i collision of the ball with the wall and before the $i+1$st collision of the ball with the object, and w_{i+1} be the velocity of the ball after the $i+1$st collision with the object and before the $i+1$st collision of the ball with the wall (Figure 11). All these velocities are measured with positive direction left to right, so w_i is positive and z_i is negative. Let $u_i = -z_i$ be the (positive) speed of the ball moving to the left before the $i+1$st collision with the object. Then, before the $i+1$st collision, the center of mass has velocity $c_i = (Mv_i + mz_i)/(M+m)$. Relative to the center of mass, the object is moving with velocity $v_i - c_i = m(v_i - z_i)/(M+m)$; the ball is moving with velocity $z_i - c_i = M(z_i - v_i)/(M+m)$. The inelastic collision leaves the velocity of the center of mass unchanged, and multiplies the velocities of the object and ball relative to the center of mass by a factor of $-\mu$.

Thus, $v_{i+1} - c_i = -\mu m(v_i - z_i)/(M+m)$; $w_{i+1} - c_i = -\mu M(z_i - v_i)/(M+m)$.

The collision of the ball with the wall multiplies its velocity by a factor of $-\mu$, so $z_{i+1} = -\mu w_{i+1}$ $= -\mu(c_i - \mu M(z_i - v_i)/(M+m))$.

Putting these together we get

$$v_{i+1} = [(M - \mu m)v_i + (1+\mu)mz_i]/(M+m) \text{ and}$$
$$z_{i+1} = \mu[-(1+\mu)Mv_i + (\mu M - m)z_i]/(M+m).$$

Rewriting in terms of u and v

$$v_{i+1} = [(M - \mu m)v_i - (1+\mu)mu_i]/(M+m) \text{ and}$$
$$u_{i+1} = \mu[(1+\mu)Mv_i + (\mu M - m)u_i]/(M+m).$$

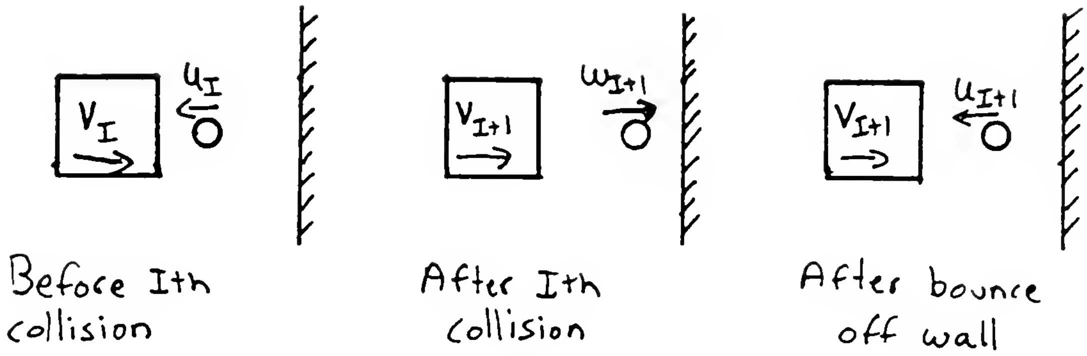


Figure 11: Ball bouncing between moving object and wall

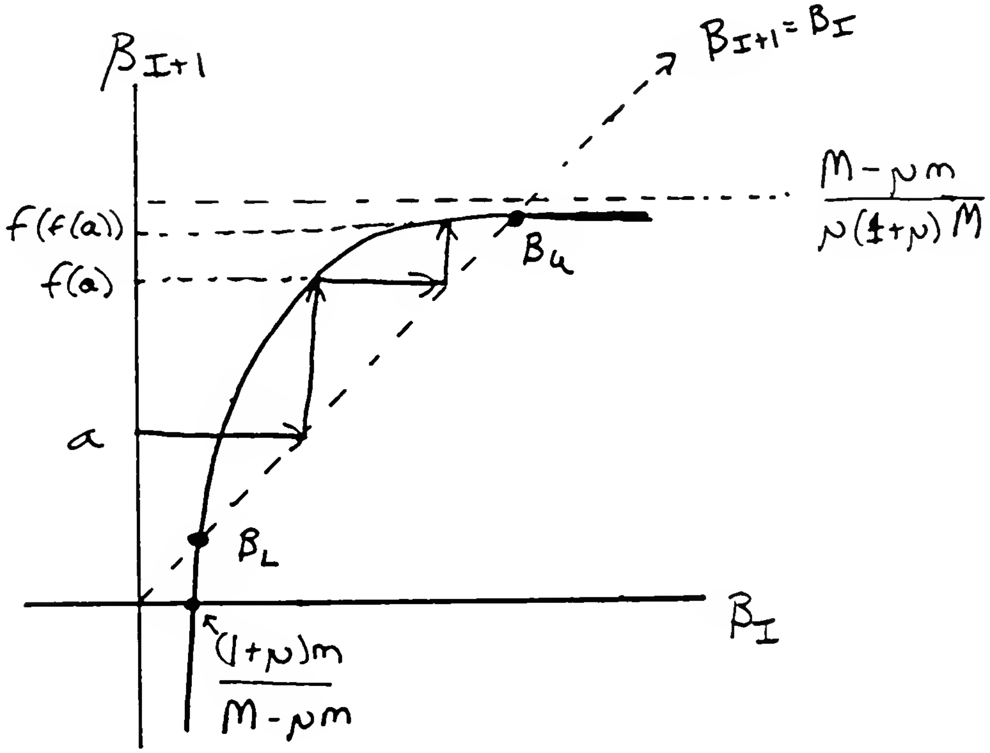


Figure 12: β_I converge to β_u.

Note that v_{i+1} continues to be positive as long as $[(M-\mu m)v_i-(1+\mu)mu_i] > 0$, or, equivalently, $v_i/u_i > (1+\mu)m/(M-\mu m)$.

The above formulas for v_{i+1} and u_{i+1} express them as linear functions of v_i and u_i, where the coefficients are determined by the constants M, m, and μ. Suppose we can find values of v_0 and u_0 such that, for some $\alpha \in (0,1)$, $v_1 = \alpha v_0$ and $u_1 = \alpha u_0$. Then, since the velocities are uniformly reduced by α, we will have $v_2 = \alpha v_1$, $u_2 = \alpha u_1$, and, in general, $v_i = \alpha^i v_0$, $u_i = \alpha^i u_0$. Thus, infinitely many bounces will occur, with the velocities approaching zero in geometric series.

We are thus looking for a solution to the equation $\alpha = v_1/v_0 = v_0/u_0$ or equivalently $v_1 u_0 = u_1 v_0$. Using the above formulas, and multiplying through by $M+m$, we get

$$[(M-\mu m)v_0 - (1+\mu)mu_0]u_0 = \mu[(1+\mu)Mv_0 + (\mu M - m)u_0]v_0$$

Simplifying, we have

$$m(1+\mu)u_0^2 + (\mu^2-1)Mu_0v_0 + \mu(1+\mu)Mv_0^2 = 0$$

$$\mu v_0^2 + (1+\mu)u_0v_0 + \frac{m}{M}\mu u_0^2 = 0$$

Let $\beta_i = v_i/u_i$. Then, solving the above equation,

$$\beta_0 = \frac{(1-\mu) \pm \sqrt{(1-\mu)^2 - 4\mu(m/M)}}{2\mu}$$

The above equation is solvable as long as the determinant $(1-\mu)^2 - 4\mu(m/M) \geq 0$; it has two roots if this inequality is strict. But this condition always holds if either μ or m/M is sufficiently small.

Thus, we have shown that, if μ is small enough, then there are two particular ratios β_L and β_U of velocities that remain constant after each iteration. We can show further that, in fact, the original ratio does not have to correspond exactly to these values; as long as the original ratio v_0/u_0 is greater than the lower of the two roots β_L, the sequence of ratio v_i/u_i will converge to the greater of the roots β_U. To show this let us return to the above equations for v_{i+1} and u_{i+1}. Dividing one equation by another, and simplifying gives us

$$\beta_{i+1} = \frac{(M-\mu m)\beta_i - (1+\mu)m}{\mu[(1+\mu)M\beta_i + (\mu M - m)}$$

This defines β_{i+1} as a function of β_i, $\beta_{i+1} = f(\beta_i)$. The graph of the function is a right hyperbole, with horizontal and vertical asymptotes, whose right-hand branch is concave downward (Figure 12). We are interested in cases where the right-hand branch intersects the line $\beta_{i+1} = \beta_i$. Now consider the sequence a, $f(a)$, $f(f(a))$ ...This sequence may be generated by starting at the point on the hyperbole, with $\beta_i = a$, drawing a horizontal line to meet the line $\beta_{i+1} = \beta_i$, and drawing a vertical line from the intersection point to the hyperbole. It is geometrically obvious and easily shown analytically that, if a starts between the two intersection points β_L and β_U, then the sequence converges upward toward β_U, while if a starts above β_U, then it converges downward toward β_U.

Finally, we must make sure that the time between bounces converges toward zero in geometric ratio. Let d_i be the "slack space" at the time of the i-th collision; that is, the distance from the object to the wall minus the diameter of the ball. Let t_i be the time between the ith and $i+1$st collisions. Between the ith and the $i+1$st collisions, the object covers a distance $d_i - d_{i+1}$ going at speed v_i; the ball goes the distance d_i from the object to the wall at speed u_i/μ and the distance $d_{i+1}$from the wall

to the object at speed u_i. Therefore, we have $t_i = (d_i - d_{i+1})/v_i = (\mu d_i + d_{i+1})/u_i$. Solving the last equation gives $d_{i+1}/d_i = (u_i - \mu v_i)/(u_i + v_i)$. In order that the times may diminish geometrically, we must ensure that $t_{i+1}/t_i \leq c < 1$. This condition is guaranteed if $d_{i+1}/d_i < cv_{i+1}/v_i$. Substituting the above equation, we get $(u_i - \mu v_i)/(u_i + v_i) < cv_{i+1}/v_i$. Assuming that v_i/u_i has converged to the value β, as discussed above, we have

$$(1 - \mu\beta)/(1 + \beta) < v_{i+1}/v_i = [(M - \mu m - (1 - \mu)m/\beta]/M + m$$

Simplifying, we get $(1 + \mu)\beta^2 M > (1 + 2\beta - \mu)m$. From above, we know that $\beta_U > (1 - \mu)/2\mu$, so this constraint will be satisfied as long as μ is sufficiently small.□

8.3 Discrete variation in real-valued functions

In this section, we define the class of piecewise Dirichlet functions, show that they give discrete variation for the states $f(x) > c$ and $f(x) = c$, and show that they are closed under arithmetical operations, exponentiation, and differentiation.

Definition 28: A *Dirichlet series* is a series of the form $\sum_{i=0}^{\infty} a_i x^{\lambda_i}$ where

i $\lambda_i < \lambda_{i+1}$ for all i;

ii. λ_i goes to infinity as i goes to infinity; and

iii. the series converges absolutely for some $x > 0$.

(Remark on standard terminology; the Dirichlet series is more usually defined in terms of a variable $u = -\log x$. Under that variable transformation, the series takes the form $\sum a_i e^{-\lambda_i u}$. Also, the standard definition of a Dirichlet series requires only convergence, not absolute convergence.)

Examples:

i. The finite series x^{-1} is a Dirichlet series.

ii. The series $x^1 + x^{\sqrt{2}} + x^{\sqrt{3}} \ldots$ is a Dirichlet series. To show that it converges absolutely in the interval [0,1), observe that, for $x < 1$, the first through third terms are less than or equal to x; the fourth through eighth terms are less than or equal to x^2 ...in general, the terms in the k^2 through $(k+1)^2 - 1$ terms are less than or equal to x^k. Therefore, the series as a whole is bounded above by $2^2 x + 3^2 x^2 + 4^2 x^3 \ldots$, which, by the ratio test, converges absolutely in the interval [0,1).

iii. The series cotangent(x)= $x^{-1} - x/3 + x^3/45 \ldots$ is a Dirichlet series.

iii. The series $\sin(1/x) = x^{-1} - x^{-3}/3! + x^{-5}/5! \ldots$ is not a Dirichlet series, since the exponents decrease rather than increase.

iv. The series $x^{1/2}/1! + x^{3/4}/2! + x^{7/8}/3! \ldots$ is not a Dirichlet series, even though it converges for all $x \geq 0$, since the exponents approach 1 rather than ∞.

Definition 29: The function $f(x)$ is Dirichlet above x_0 if there is a Dirichlet series

$$\sum_{i=0}^{\infty} a_i (x - x_0)^{\lambda_i}$$

that converges absolutely to $f(x)$ in some open interval (x_0, x_1). $f(x)$ is Dirichlet below x_0 if there is a Dirichlet series

$$\sum_{i=0}^{\infty} a_i(x_0 - x)^{\lambda_i}$$

that converges absolutely to $f(x)$ in some open interval (x_1, x_0).

Lemma 30: If $f(x)$ is analytic at x_0, then $f(x)$ is Dirichlet above and below x_0.

Proof: Immediate from the definitions.□

Theorem 31: If $f(x)$ is Dirichlet above x_0, with a series that converges in the interval (x_0, x_1), then $f(x)$ is analytic in the interval (x_0, x_1). (Note: the converse does not hold.)

Proof: Let x_c be any point in (x_0, x_1). Each term of the Dirichlet series $a_i(x - x_0)^{\lambda_i}$ is analytic in the neighborhood of x_c and therefore can be written as a power series $\sum d_{ij}(x - x_c)^j$ that converges absolutely in the neighborhood of x_c. We can therefore write

$$f(x) = \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} d_{ij}(x - x_c)^j) = \sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} d_{ij})(x - x_c)^j$$

Since the first sum above is the absolutely convergent sum of absolutely convergent series, the rearrangement of the order is correct.□

Definition 32: A function $f(x)$ is Dirichlet in the open interval (x_0, x_1) if f is Dirichlet above x_0, Dirichlet below x_1, and analytic throughout (x_0, x_1). The function $f(x)$ is Dirichlet in an interval I that is half-open or closed if $f(x)$ is Dirichlet in some open interval containing I.

Example: The function $f(x) = 1/x(x-1)$ is Dirichlet in the interval (0,1).

Definition 33: Let $f_1, f_2 \ldots$ be a collection of functions and let P be a mapping such that

i. The conditions of definition 8 are satisfied.

ii. $f_i(x)$ is Dirichlet in the interval $P(f_i)$.

Then the splice of P is said to be *piecewise Dirichlet.*

Observation: Any Dirichlet series can be rewritten in normalized form $Cx^{\beta}(1+c_1x^{\alpha_1}+c_2x^{\alpha_2}\ldots)$ where $0 < \alpha_1 < \alpha_2 < \ldots$ by factoring out the leading term. We will call Cx^{β} the leading factor, and call $1 + c_1x^{\alpha_1} + c_2x^{\alpha_2}\ldots$ the series factor. It is easily shown that the series factor approaches 1 as x goes to zero.

Lemma 34: If $f(x)$ is Dirichlet above x_0, then there is an interval (x_0, x_1) in which $f(x)$ is either always positive, always negative, or always zero.

Proof: Put the Dirichlet series in normalized form, as above. Since the series factor converges to 1, we may choose an ϵ small enough that $1 + c_1t^{\alpha_1} + c_2t^{\alpha_2}\ldots$ is greater than 0 for all $t \in (0, \epsilon)$. Then, for $x \in (x0, x0 + \epsilon)$, the series factor is always positive, and the leading factor always has the sign of the constant coefficient C, so $f(x)$ has the same sign as C throughout $(x0, x0 + \epsilon)$.□

Theorem 35: If $f(x)$ is piecewise Dirichlet, then it has discrete variation.

Proof: Immediate from Lemma 34 and Theorem 5.□

Lemma 36: If $f(x)$ and $g(x)$ are both Dirichlet above x_0, or both are Dirichlet below x_0, then so are $f(x) + g(x)$, $f(x) - g(x)$, and $f(x) \cdot g(x)$. The functions $g(x)/f(x)$, is Dirichlet above (below) x_0 unless $f(x)$ is identically zero in an interval above (below) x_0. The function $f(x)^{\alpha}$ is Dirichlet

above (below) x_0 if $f(x_0) \geq 0$ and $f(x)$ is not identically zero in an interval above (below) x_0. The derivative $f'(x)$ is Dirichlet.

Proof: In each case, the new formal series is easily calculated from the old series, and absolute convergence follows from standard theorems.

Addition and subtraction: Add/subtract term by term. If two series converge absolutely, then their sum does as well. [Knopp, p. 138]

Multiplication: Multiply term by term. If two series converge absolutely, then their product does as well in the smaller of the two intervals of convergence. [Knopp, p. 146]

Exponentiation: Express $f(x)$ in normal form: $f(x) = Cx^{\beta}(1 + c_1x^{\alpha_1} + c_2x^{\alpha_2}\ldots)$ where $C \neq 0$. Suppose that f converges absolutely in the interval $(0, r)$. We have

$$f^q(x) \ = C^q \ x^{q\beta} \ (1 + c_1x^{\alpha_1} + c_2x^{\alpha_2}\ldots)^q$$

Let $z = (c_1x^{\alpha_1} + c_2x^{\alpha_2}\ldots)$ Note that z goes to 0 as x goes to zero. Then, by the binomial theorem, the third factor above has the form

$$(1 + z)^q = 1 + qz + \frac{q(q-1)}{2!}z^2 + \ldots$$

Moreover, this series converges absolutely for $\mid z \mid< 1$ [Knopp, p. 208]. Since $z(x)$ goes to zero when x goes to zero, this condition holds for small enough z.

By the remark on multiplication above, each power

$$z(x)^k = (c_1x^{\alpha_1} + c_2x^{\alpha_2}\ldots) \cdot \ldots \cdot (c_1x^{\alpha_1} + c_2x^{\alpha_2}\ldots) \ (k \text{ times})$$

can be expanded as a series in x, which converges absolutely within the same radius as $z(x)$. Substituting these in the above series, we have the term factor of $f(x)^q$ expressed as a sum of terms each of which is a power series in x. Two points remain to be established:

1. If x is chosen small enough so that the series for $z(x)$ converges absolutely to a value less than 1, then the overall series is the absolutely convergent sum of a collection of absolutely convergent series, and hence converges absolutely.
2. We must show that the series can be rearranged into descending powers of x that go to minus infinity. An equivalent condition is that for any M, there are only finitely many terms x^{δ} where $\delta < M$. This is immediate, since the smallest exponent in the expansion of $z(x)^k$ is $k\alpha_1$. Hence, powers of x that are less than M are generated only from series earlier than $z(x)^{M/\alpha_1}$.□

Quotient: Rewrite $g(x)/f(x)$ as $g(x)f(x)^{-1}$ and apply the above results on multiplication and exponentiation.

Differentiation: Differentiate term by term. Convergence is immediate from the ratio test.□

Theorem 37: If $f(x)$ and $g(x)$ are piecewise Dirichlet, then then so are $f(x)+g(x)$, $f(x)-g(x)$, $f(x) \cdot g(x)$. The functions $g(x)/f(x)$, and $f(x)^a$ are piecewise Dirichlet unless $f(x)$ is identically zero over some finite interval. The derivative $f'(x)$ is piecewise Dirichlet, assuming some convention for defining the derivative at points where the function is not differentiable.

Proof: Immediate from lemma 36 and theorem 6.□

Still, there are well-behaved functions, like $\log(x)$ or e^{-1/x^2}, that are not representable by Dirichlet series at $x = 0$. It can be shown that there are larger classes of functions that contains these, as well as all the piecewise Dirichlet functions; that is closed under the operations of theorem 35; and that is guaranteed to vary discretely. However, the description of this set and proof of these properties is so long and of such limited interest that I have not included them here.

8.4 Spatial Fluents

In this section we prove theorem 15 from section 4.4. Informally, the theorem states that, in a situation where all objects have semi-algebraic shapes and piecewise analytic motions, any Boolean fluent that can be stated in an algebraic language varies discretely. The formal statement of this result is given as theorem 44 below. Viewed from the proper perspective, this result is an almost trivial consequence of a classic result of Tarski [1951], stating that the solution space for any open formula in an algebraic language is a semi-algebraic set (theorem 42 below), combined with theorem 17. This section, thus, consists of setting up that perspective.

A position of an object in n-dimensional space can be characterized in terms of a mapping from $\Re^n$ to $\Re^n$. (If the object is rigid, then the mapping is orthonormal.) A motion of the object through time is therefore a function from time to mappings, or, in other words, a mapping from $\Re^1 \times \Re^n$ to $\Re^n$. It will be convenient to consider motions that are governed by a collection of real-valued parameters, $p_1 \dots p_k$. Each of these parameters will eventually be viewed as a function of time, but initially it will be useful to consider them as independently varying parameters. We can therefore adopt the following definition:

Definition 38: A *k-parameter motion in n-dimensional space* is a mapping from $\Re^k \times \Re^n$ to $\Re^n$. If M is such a mapping, we write $M(p_1 \dots p_k, \vec{x}) = \vec{y}$

Definition 39: A mapping M from $\Re^a$ to $\Re^b$ is *algebraic* if each coordinate in $M(\vec{x})$ is an algebraic function (multinomial) of the coordinates of $\vec{x}$. (It is easily shown that whether or not a mapping is algebraic is independent of the coordinate system.

Definition 39 is applied to k-parameter motions by taking a to be $k+n$ and b to be n.

Example: Let $n = 3$ and $k = 9$. The k-parameter motion

$$M(\vec{x}) = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is algebraic.

For any fixed value of the parameters, a k-parameter motion defines a function from $\Re^n$ to $\Re^n$. This is extended in the natural way to define a mapping from n-dimensional regions to n-dimensional regions. If $\mathbf{R}$ is a region in $\Re^n$, and M is a k-parameter motion, we write "$M(p_1 \dots p_k, \mathbf{R})$" to mean the image of $\mathbf{R}$ under the mapping with the particular parameter values $p_1 \dots p_k$.

Definition 40: The *language of real algebra* is the first-order language with equality and with the non-logical primitives '0', '1', 'plus(X, Y)', 'times(X, Y)', and 'greater(X, Y)', where the domain is taken to be the real numbers and the primitives are given their usual interpretation.

Definition 41: Let $\mathbf{R}_1 \dots \mathbf{R}_m$ be m regions in $\Re^n$. A *quantification over* $\mathbf{R}_i$ is a restricted quantification of the form $Q_{<x_1 \dots x_k> \in \mathbf{R}_i}$, where Q is either the universal or existential quantifier. The property $\Phi(\mathbf{R}_1 \dots \mathbf{R}_m)$ is an *algebraic property* of the regions if it is equivalent to a formula written in the language of real algebra, together with quantifications over the $\mathbf{R}_i$.

Examples:

i. The property "**R** and **S** overlap" is algebraic, since it can be written (in two dimensions) in the form

$$\exists_{<W,X> \in \mathbf{R}} \exists_{<Y,Z> \in \mathbf{S}}\ W = Y \land X = Z$$

ii. The property "The minimum distance from **R** to **S** is equal to D," is algebraic, since it can be written (in two dimensions) as

$[\forall_{<W,X>\in \mathbf{R}} \forall_{<Y,Z>\in \mathbf{S}} (W-Y)\cdot(W-Y)+(X-Z)\cdot(X-Z) \geq D\cdot D] \wedge$
$[\forall_{E>D} \exists_{<W,X>\in \mathbf{R}} \exists_{<Y,Z>\in \mathbf{S}} (W-Y)\cdot(W-Y)+(X-Z)\cdot(X-Z) < E\cdot E]$
(The subtractions above can easily be expressed in terms of plus.)

iii. The property "The volume of **R** is greater than 3," is not an algebraic property. Note that the basic definition of volume involves an integral, which is not an algebraic operation.

We now invoke the classic theorem of Tarski's on the solvability of real algebra.

Theorem 42: Let ϕ be a formula in the language of real algebra with k free variables $\alpha_1 \dots \alpha_k$. Let $\vec{\alpha} = <\alpha_1 \dots \alpha_k>$ be a k-tuple of values for the α_i. The solution space of ϕ — that is, the set of values of $\vec{\alpha}$ for which ϕ is true — is a semi-algebraic region in $\Re^k$.

Proof: See [Tarski, 51].

The following corollary is just a special case of Tarski's theorem.

Corollary 43: Let $\mathbf{R}_1 \dots \mathbf{R}_q$ be q fixed semi-algebraic regions in $\Re^n$. Let $M_1 \dots M_q$ be q different algebraic parameterized motions. Let $\alpha_1 \dots \alpha_k$ be the parameters of all the M_i. Let Φ be an algebraic property of q regions. Thus the formula $\Phi(M_1\mathbf{R}_1 \dots M_q\mathbf{R}_q)$ is an open formula whose free variables are the α_i. The solution set of this open formula is a semi-algebraic region in $\Re^k$.

Proof: Since each region $\mathbf{R}_i$ and each motion M_i is algebraic, the restricted quantification over the range $< x_1 \dots x_n > \in M(p_{i1} \dots p_{ik}, \mathbf{R}_i)$ can be expressed as a quantification over the x_i, subject to an algebraic constraint over the x_i and p_j. Thus, the whole condition Φ can be expressed as a formula in the language of real algebra with the parameters as free variables. By Tarski's theorem, the solution space of this open formula is a semi-algebraic region.□

Theorem 44: As in corollary 42, let $\mathbf{R}_1 \dots \mathbf{R}_q$ be q semi-algebraic regions in $\Re^n$; let $M_1 \dots M_q$ be q different algebraic k-parameter motions; let $\alpha_1 \dots \alpha_k$ be the combined parameters of the M_i; and let Φ be an algebraic property of q regions. Furthermore assume that each of the k parameters is a piecewise analytic function of time. Then the fluent "Φ holds on $M_1(t)(\mathbf{R}_1) \dots M_q(t)(\mathbf{R}_q)$" varies discretely.

Proof: Consider the configuration space $\Re^z$ of parameter values. By corollary 41, the set of values satisfying Φ is semi-algebraic. Let $\vec{x}(t)$ be the "configuration point" of the tuple of all the parameter values over time. Clearly, since each parameter is semi-analytic, so is $\vec{x}(t)$. Applying theorem 17, we conclude that the truth of Φ varies discretely.□

Example: Let O_1 be an object that occupies region R_1 at time $t = 0$ and that is rotating around point $\vec{c_1}$ with constant angular velocity ω_1. Define O_2, R_2, c_2, ω_2 be likewise. Assume that R_1 and R_2 are semi-algebraic. Then the motion of O_1 is characterized by the mapping $M_1(t, \vec{x}) = \vec{c_1} + A_1(t)(\vec{x} - \vec{c_1})$ where $A_1(t)$ is the matrix

$$A_1(t) = \begin{bmatrix} \cos(\omega_1 t) & -\sin(\omega_1 t) \\ \sin(\omega_1 t) & \cos(\omega_1 t) \end{bmatrix}$$

. The motion of O_2 is analogous. We now define the four parameters $\alpha_1(t) = \sin(\omega_1 t)$, $\alpha_2(t) = \cos(\omega_1 t)$, $\alpha_3(t) = \sin(\omega_2 t)$, $\alpha_4(t) = \cos(\omega_2 t)$. Since M is an algebraic mapping over the α_i and $\vec{x}$, we can apply theorem 44, and infer that states such as "O_1 overlaps O_2" vary discretely. Note that, since the mapping M is a trigonometric function of t, we cannot apply corollary 43 directly to the parameter t.

9 References

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